Possible Characteristic Operators of a Group.

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§ 1. *Introduction.*

A characteristic operator of a group G is an operator which corresponds to itself in every possible automorphism of G. Every possible group contains at least one characteristic operator, viz., the identity. A necessary and sufficient condition that an abelian group involves another characteristic operator is that it contains one and only one operator of order 2. Hence we may confine our attention to non-abelian groups in the study of possible characteristic operators. In what follows we shall therefore assume that G is non-abelian.

The characteristic operators of G clearly constitute an abelian characteristic subgroup of G. The main object of the present article is to prove that it is possible to construct a non-abelian group which has for its characteristic operators all the operators of an arbitrary abelian group. By means of the fact that at least one such non-abelian group can be constructed, it is easy to prove that the number of such groups is always infinite. In fact, such an infinite system can be obtained directly by forming the direct products of one such group and simple groups of composite orders.

§ 2. *Group Containing a Characteristic Operator Whose Order is an Arbitrary Power of a Prime Number.*

Let p represent any prime number of the form $1+kq^{\alpha}$, α being an arbitrary positive integer. The transitive substitution group K of degree p and of order $p(p-1)$ contains a subgroup of order pq^{α} which involves no invariant operator except identity. To obtain the desired group in the form of a regular substitution group, we shall first construct an intransitive substitution group H by establishing a (1, 1) correspondence between q^{α} regular cyclic groups of order pq^{β}, $\beta \overline{\gtrless} \alpha$. The q^{α} transitive constituents of H are conjugate under the powers of a substitution t of order q^{α} which is commutative with every substitution of H, q being a prime number.

Let s be a substitution of order q^α which is commutative with t and with the substitutions of order q^β contained in H, but transforms the substitutions of order p contained in H into a power which belongs to exponent q^α mod p. That is, the group generated by s and a substitution of order p contained in H is simply isomorphic with K, and we may assume that the group generated by s and H involves q^α transitive constituents on the letters of H. The product of st and a substitution of order q^β in one of the transitive constituents of H is of order $q^{\alpha+\beta}$, and the q^α-*th* power of this product is the substitution of order q^β contained in H.

The group G generated by H and this product has for its central the cyclic group of order q^β, and the central quotient group of G is simply isomorphic with K. Since G involves only one subgroup of order p, and since all the operators of G which transform one of the generating operators of this subgroup into its n-*th* power must also transform each of its other operators into the same power, it results that in every possible automorphism of G the operators which transform its operators of order p into a particular power must correspond to themselves. In particular, the operators of order $q^{\alpha+\beta}$ in G which transform each of its operators of order p into the same power must correspond to themselves in each of these automorphisms.

The operators of order $q^{\alpha+\beta}$ in G whose q^α-*th* power is the same operator of order q^β in H clearly transform the operators of order p in H into one or more powers, which are distinct from the powers into which these operators of order p are transformed by those operators of order $q^{\alpha+\beta}$ whose q^α-*th* power is another operator of order q^β in H. From this it results directly that the operators of order q^β contained in the central of G are characteristic operators of G. Since these operators generate the central of G, this central must be composed of characteristic operators of G.

Since q is an arbitrary prime number, and β is an arbitrary positive integer, we have established the lemma that *it is possible to construct a non-abelian group which has an operator of any desired prime power order as a characteristic operator*. In fact, the existence of such a characteristic operator of order q^β is independent of p and α provided they satisfy the stated conditions, and hence it results from the preceding arguments that any one of a multiply infinite number of different groups may be used for G.

§ 3. *General Case.*

To prove that it is possible to construct a group G which has all the operators of an arbitrary abelian group G' as characteristic operators, it is

only necessary to prove that a set of independent generators of G' are characteristic operators of a certain group G. We shall first consider the case when the order of each of the independent generators of G' is a power of q, and we shall let q^β represent one of the largest of these orders.

Let $s_1, s_2, \ldots, s_r$ represent a set of independent generators of G' and find r distinct prime numbers $p_1, p_2, \ldots, p_r$ of the form $1+kq^\beta$. According to the preceding section we can construct r distinct groups $G_1, G_2, \ldots, G_r$ such that the characteristic operators of these groups are generated by $s_1, s_2, \ldots, s_r$, respectively, and that the prime numbers of the form $1+kq^\beta$ which divide their orders are $p_1, p_2, \ldots, p_r$, respectively. The orders of the r groups $G_1, G_2, \ldots, G_r$ may be assumed to be

$$p_1q^\beta,\ p_2q^\beta,\ \ldots,\ p_rq^\beta$$

multiplied by the orders of $s_1, s_2, \ldots, s_r$, respectively.

The direct product of $G_1, G_2, \ldots, G_r$ contains a characteristic cyclic subgroup of order $p_1, p_2, \ldots, p_r$. In any automorphism of this direct product G_1 corresponds to a subgroup whose operators are commutative with each of the operators in the cyclic subgroup of order $p_2, p_3, \ldots, p_r$ contained in this direct product. To the operators of order $q^{\alpha'+\beta}$ in G_1, α' being the order of s_1, there must therefore correspond operators whose q^β power is the same as the q^β power of the corresponding operators of G_1. Hence s_1 is a characteristic operator of this direct product. As s_1 is an arbitrary independent generator of G', the direct product under consideration has all the operators of G' for its characteristic operators.

When the order of G' is not a power of a prime number, G' is the direct product of characteristic subgroups whose orders are such powers. By constructing groups whose characteristic operators constitute these characteristic subgroups, and then forming the direct product of these groups, we obtain a group whose characteristic operators are the operators of G'. This completes a proof of the theorem

It is possible to construct non-abelian groups whose characteristic operators constitute an arbitrary abelian group.